

# The inverse loop transform

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## Abstract

The loop transform in quantum gauge field theory can be recognized as the Fourier transform (or characteristic functional) of a measure on the space of generalized connections modulo gauge transformations. Since this space is a compact Hausdorff space, conversely, we know from the Riesz-Markov theorem that every positive linear functional on the space of continuous functions thereon qualifies as the loop transform of a regular Borel measure on the moduli space.

In the present article we show how one can compute the finite joint distributions of a given characteristic functional, that is, we derive the inverse loop transform.

## 1 Introduction

Recently, there has been made considerable progress in the development of a rigorous calculus on the space of generalized (that is, distributional) connections modulo gauge transformations  $\overline{\mathcal{A}/\mathcal{G}}$  for quantum gauge field theories based on compact gauge groups [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. These developments can be summarized roughly as follows :

- The space of histories for quantum gauge field theory arises as the Gel'fand spectrum of the  $C^*$  algebra generated by the Wilson loop functionals [1].
- There are two equivalent, useful descriptions of that spectrum :
  - a) it can be recognized as the set of *all* homomorphisms from the group of loops into the gauge group [2].
  - b) it arises as the projective limit of the measurable spaces defined by restricting the homomorphisms to the cylindrical subspaces defined by piecewise analytical graphs  $\gamma$  [5, 6].
- The integral calculus on  $\overline{\mathcal{A}/\mathcal{G}}$  is governed by the fact that the spectrum is always a compact Hausdorff space so that regular ( $\sigma$ -additive) Borel measures thereon are in one to one correspondence with positive linear functionals on  $C(\overline{\mathcal{A}/\mathcal{G}})$  [1]. Interestingly, even diffeomorphism invariant measures can be constructed thereon [2, 3].

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Furthermore, the contact with constructive quantum gauge field theory is made by the so-called loop transform which is nothing else than the characteristic functional of the given measure [10, 11], the role of the usual Bochner theorem [15] being played by the Riesz-Markov theorem.

- Even differential geometry can be developed on cylindrical subspaces of  $\overline{\mathcal{A}/\mathcal{G}}$  [7, 8, 9].
- The motivation of the authors involved in these developments comes from quantum gravity formulated as a dynamical theory of connections [16]. The mathematical progress made has given rise to some new results [12] in quantum gravity. While this is a theory of complex-valued connections, contact with the above mentioned results and techniques can be made by a coherent state transform [13]. The correct version of that transform was found in [14] for the case of pure gravity and extended in [14a] to incorporate matter.

In this paper we make yet another contribution to this subject, namely we construct the analogue of the inverse Fourier transform on  $\mathbb{R}^n$  [17] which we call the “inverse loop transform”. This will allow us to reconstruct a measure as defined by its finite joint distributions from its characteristic functional. In particular, given a (singular) knot invariant satisfying certain additional conditions, we can find out to what measure it corresponds.

The plan of the paper is as follows :

In section 2 we recall the basic notions from calculus on  $\overline{\mathcal{A}/\mathcal{G}}$  as far as necessary for the present context. The interested reader is referred to the article [12] for further details.

In section 3 we first formulate and prove the analogue of the inverse Fourier transform theorem for compact gauge groups and then give a new definition of the loop transform based on the notion of “loop networks” [18, 19]. This notion will prove useful in proving the inverse loop transform.

In section 4 we conclude by displaying the inverse loop transform of physically interesting characteristic functionals.

In an appendix we indicate how all the results of this paper can be rewritten in terms of edges rather than loops which is sometimes more convenient in applications beyond integration on  $\overline{\mathcal{A}/\mathcal{G}}$ . In particular we introduce the notion of an “edge-network” which generalizes the notion of a “spin-network” [18, 19] to an arbitrary compact gauge group and also allows us to give a closed and compact expression for a spin-network state without referring to a graphical notation [18].

## 2 Preliminaries

We give here only the absolutely necessary information in order to fix the notation. For further details see [12] and references therein.

The space  $\overline{\mathcal{A}/\mathcal{G}}$  of generalized connections modulo gauge transformations is the Gel’fand spectrum of the Abelian  $C^*$  algebra generated by the Wilson-loop functionals for smooth connections, that is, traces of the holonomy for piecewise analytic loops in the base manifold  $\Sigma$ . As such it is a compact Hausdorff space and therefore measures on that space are in one to one correspondence with positive linear

functionals on  $C(\overline{\mathcal{A}/\mathcal{G}})$ .

A certain natural measure  $\mu_0$  will play a very crucial role in this article so that we go now into more details :

In what follows,  $\gamma \subset \Sigma$  will always denote a finite, unoriented, piecewise analytic graph, meaning that it is the union of a finite number of analytic edges and vertices. Its fundamental group  $\pi_1(\gamma)$  is then finitely generated by some loops  $\beta_1(\gamma), \dots, \beta_{n(\gamma)}(\gamma)$  which we fix once and for all together with some orientation and which are based at some arbitrary but fixed basepoint  $p \in \Sigma$ ,  $n(\gamma) := \dim(\pi_1(\gamma))$  being the number of independent generators of the fundamental group of  $\gamma$ . A function  $f$  on  $\overline{\mathcal{A}/\mathcal{G}}$  is said to be cylindrical with respect to a graph  $\gamma$ ,  $f \in \text{Cyl}_\gamma(\overline{\mathcal{A}/\mathcal{G}})$ , if it is a function only of the finite set of arguments  $p_\gamma(A) := (h_{\beta_1}(A), \dots, h_{\beta_n}(A))$  where  $h_\alpha(A)$  is the holonomy of  $A$  along the loop  $\alpha$ . A measure  $\mu$  is now specified by its finite joint distributions  $\mu_\gamma$  which are defined by

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu(A) f(A) = \int_{G^n} d\mu_\gamma(g_1, \dots, g_n) f_\gamma(g_1, \dots, g_n) \quad (2.1)$$

where  $f = f_\gamma \circ p_\gamma$  and  $f_\gamma : G^n \rightarrow \mathbb{C}$  is a gauge invariant function. In order that this definition makes sense we have to make sure that if we write  $f = f_\gamma \circ p_\gamma = f_{\gamma'} \circ p_{\gamma'}$  in two different ways as a cylindrical function where  $\gamma \subset \gamma'$  is a subgraph of  $\gamma'$ , then we should have that the so-called consistency conditions

$$\int_{G^n} d\mu_\gamma f_\gamma = \int_{G^{n'}} d\mu_{\gamma'} f_{\gamma'} \quad (2.2)$$

are satisfied.

The natural measure  $\mu_0$  is the induced Haar measure, meaning that  $d\mu_{0,\gamma}(g_1, \dots, g_n) = d\mu_H(g_1) \dots d\mu_H(g_n)$ . One can check that the consistency conditions are satisfied [2] and that the so defined cylindrical measure has a  $\sigma$ -additive extension  $\mu_0$  on the projective limit measurable space of the family of measurable spaces  $\overline{\mathcal{A}/\mathcal{G}}_\gamma$  [5]. The space  $\overline{\mathcal{A}/\mathcal{G}}_\gamma$  is defined to be the set of all homomorphisms from the group of based loops restricted to  $\gamma$  into the gauge group modulo conjugation while  $\overline{\mathcal{A}/\mathcal{G}}$  is the set of all homomorphisms from the whole loop group into  $G$  modulo conjugation. Note that the semi-group of loops with respect to compositions of loops can be given a group structure by identifying paths that are traversed in the opposite direction with the inverse of the original path.

### 3 The inverse loop transform

#### 3.1 The inverse Fourier transform for compact groups

Let us recall some basic facts from harmonic analysis on compact gauge groups [20].

**Definition 3.1** *Let  $\{\pi\}$  denote the set of all finite dimensional, non-equivalent (we fix one representant from each equivalence class once and for all), unitary, irreducible representations of the compact gauge group  $G$ , let  $d_\pi$  be the dimension of  $\pi$  and let*

$\mu_H$  be the normalized Haar measure on  $G$ .

For any  $f \in L_1(G, d\mu_H)$  define the Fourier transform of  $f$  by

$$\hat{f}_\pi^{ij} := \int_G d\mu_H(g) \sqrt{d_\pi} \pi_{ij}(g) f(g), \quad i, j = 1, \dots, d_\pi \quad (3.1)$$

where  $\pi_{ij}(g)$  denotes the matrix elements of  $\pi(g)$ .

Note that this definition makes sense because the matrix elements of  $\pi(g)$  are bounded by 1.

**Definition 3.2** The Fourier transform of a function is said to be  $\ell_1$  or  $\ell_2$  respectively iff

$$\|\hat{f}\|_1 := \sum_\pi \sum_{i,j=1}^{d_\pi} \sqrt{d_\pi} |f_\pi^{ij}| < \infty \text{ or } \|\hat{f}\|_2 := \sum_\pi \sum_{i,j=1}^{d_\pi} |f_\pi^{ij}|^2 < \infty. \quad (3.2)$$

The Fourier series associated with a function  $f$  on  $G$  such that  $\hat{f} \in \ell_1$  is given by

$$\tilde{f}(g) := \sum_\pi \sum_{i,j=1}^{d_\pi} \hat{f}_\pi^{ij} \pi_{ij}(g) \sqrt{d_\pi}. \quad (3.3)$$

The analogue of the Plancherel theorem for  $\mathbb{R}^n$  is the Peter&Weyl theorem

**Theorem 3.1 (Peter&Weyl)** 1) The functions  $g \rightarrow \sqrt{d_\pi} \pi_{ij}(g)$ ,  $i, j = 1, \dots, d_\pi$  form a complete and orthonormal system on  $L_2(G, d\mu_H)$ .

2) For any  $f \in L_2(G, d\mu_H)$  it holds that  $f = \tilde{f}$  in the sense of  $L_2$  functions and the Fourier transform is a unitary map  $\wedge : L_2(G, d\mu_H) \rightarrow \ell_2$ .

The author was unable though to find the analogue of the inverse Fourier transform for compact groups in the literature which we therefore prove here. This theorem answers the question whether a function which is only  $L_1$  can be represented, in the  $L_1$  sense, by its Fourier transform.

**Theorem 3.2** Let  $f \in L_1(G, d\mu_H)$  such that also  $\hat{f} \in \ell_1$ . Then  $f(g) = \tilde{f}(g)$  on  $L_1$ .

Proof :

Let be given any  $\hat{k} \in \ell_1$ . Then the Fourier series

$$\tilde{f}_k := \sum_{\pi, i, j} [\sum_k f_\pi^{ij} \bar{k}_\pi^{jk}] \pi_{ij}(g) \sqrt{d_\pi} \quad (3.4)$$

still converges absolutely as can be seen from the following considerations : upon using the Schwarz inequality we obtain the estimate

$$|\sum_k f_\pi^{ik} \bar{k}_\pi^{jk}| \leq \sum_k |f_\pi^{ik}| |k_\pi^{jk}| \leq \sqrt{\sum_k |f_\pi^{ik}|^2} \sqrt{\sum_l |k_\pi^{jl}|^2} \leq [\sum_k |f_\pi^{ik}|] [\sum_l |k_\pi^{jl}|]$$

so that

$$\|\tilde{f}_k\|_1 \leq \sum_\pi [\sum_{i,k} |f_\pi^{ik}| \sqrt{d_\pi}] [\sum_{j,l} |k_\pi^{jl}| \sqrt{d_\pi}] \leq [\sum_{\pi, i, k} |f_\pi^{ik}| \sqrt{d_\pi}] [\sum_{\sigma, j, l} |k_\sigma^{jl}| \sqrt{d_\sigma}] = \|\hat{f}\|_1 \|\hat{k}\|_1. \quad (3.5)$$

Hence (3.4) is well-defined and we may write

$$\begin{aligned}
\tilde{f}_k(g) &= \sum_{\pi,j,k} \sqrt{d_\pi} \bar{k}_\pi^{jk} \left[ \int d\mu_H(h) \pi(h^{-1}g)_{kj} f(h) \right] \\
&= \sum_{\pi,j,k} \bar{k}_\pi^{jk} \sqrt{d_\pi} \left[ \int d\mu_H(h) \pi(h^{-1})_{kj} (L_g f)(h) \right] \\
&= \int d\mu_H(h) (L_g f)(h) \sum_{\pi,j,k} \sqrt{d_\pi} \bar{k}_\pi^{jk} \pi_{jk}(h) \\
&= \int d\mu_H(h) (L_g f)(h) \tilde{k}(h) .
\end{aligned} \tag{3.6}$$

In the second line we have made use of the translation invariance of the Haar measure and the unitarity of the representation,  $(L_g f)(h) = f(gh)$  is the definition of the left regular representation of  $G$ , in the third line we could switch integration and summation because  $\hat{k} \in \ell_1$  and  $\bar{\pi}_{jk} L_g f \in L_1(G, d\mu_H)$  and finally in the last step we have used the definition of the Fourier series (3.3).

We now choose  $k_\pi^{ij} := \sqrt{d_\pi} \pi_{ij}(1) e^{-t\lambda_\pi}$  where  $\lambda_\pi$  are the eigenvalues of the Casimir operator  $\Delta$  in the representation  $\pi$ . Then  $\tilde{k}(g) = \rho_t(g)$  becomes the heat kernel on  $G$  [21], that is, the fundamental solution of the equation

$$\left[ \frac{\partial}{\partial t} - \Delta \right] \rho_t(g) = 0, \quad \rho_0(g) = \delta_{\mu_H}(g, 1) . \tag{3.7}$$

The motivation for this choice is of course that for  $t \rightarrow 0$  the lhs of (3.6) tends to  $\tilde{f}(g)$  while the rhs should tend to  $f(g)$ . Indeed, since  $\rho_t$  tends to the  $\delta$  distribution on  $G$  wrt the Haar measure, this would be straightforward to see if  $f \in C^\infty(G)$ . To show that this is true even for  $f \in L_1(G, d\mu_H)$  we argue as follows : denote  $(f \star \rho_t)(g) := \int d\mu_H(h) (L_g f)(h) \rho_t(h)$  then we have

$$\begin{aligned}
\|f \star \rho_t - f\|_1 &= \int d\mu_H(g) \left| \int d\mu_H(h) [R_h f - f](g) \rho_t(h) \right| \\
&\leq \int d\mu_H(g) \int d\mu_H(h) |R_h f - f|(g) \rho_t(h)
\end{aligned} \tag{3.8}$$

where we have used the normalization and positivity of the heat kernel and  $(R_h f)(g) = f(gh)$ .

The idea is now to split the integration domain of the inner integral into a compact neighbourhood  $U$  of the identity of  $\mu_H$  volume  $\delta$  and its complement  $G - U$  in  $G$  :

$$\begin{aligned}
\|f \star \rho_t - f\|_1 &\leq \int_G d\mu_H(g) \int_U d\mu_H(h) |R_h f - f|(g) \rho_t(h) \\
&\quad + \int_G d\mu_H(g) \int_{G-U} d\mu_H(h) |R_h f - f|(g) \rho_t(h) =: I + II .
\end{aligned} \tag{3.9}$$

Since the heat kernel gets concentrated at the identity for  $t \rightarrow 0$  we can estimate the second integral while we will estimate the first integral by a compactness argument. The details are as follows. We first assume that  $f \in C(G)$ . This assumption will be dropped later again. Then  $f$  is uniformly continuous on the compact set  $gU = \{R_h g; h \in U\}$  and therefore there is a non-negative function  $\omega(\delta)$ ,  $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$  such that  $|R_h f - f|(g) \leq \omega(\delta) \forall h \in U$ . Accordingly

$$I \leq \omega(\delta) \int_U d\mu_H(h) \rho_t(h) \leq \omega(\delta) \tag{3.10}$$

due to the normalization of the heat kernel.

The continuous function  $(g, h) \rightarrow (R_h f - f)(g) \rho_t(h)$  is measurable on  $G \times G$  wrt the Borel measure  $\mu_H \times \mu_H$  (recall [21] that  $\rho_t$  is even real analytic). Since  $|\int_G d\mu_H(h) \rho_t(h) \int d\mu_H(g) (R_h f - f)(g)| \leq 2\|f\|_1 < \infty$  the theorem of Fubini allows us to switch the integrations in the second integral and we arrive at

$$\begin{aligned} II &= \int_{G-U} d\mu_H(h) \rho_t(h) \int_G d\mu_H(g) |R_h f - f|(g) = \int_{G-U} d\mu_H(h) \rho_t(h) \|R_h f - f\|_1 \\ &\leq 2\|f\|_1 \int_{G-U} d\mu_H(h) \rho_t(h). \end{aligned} \quad (3.11)$$

Now for any  $\epsilon > 0$  we find a  $\delta(\epsilon)$  such that  $\omega(\delta(\epsilon)) < \epsilon/2$  and for this so chosen  $\delta(\epsilon)$  we find a  $t(\epsilon, f)$  so that  $2\|f\|_1 \int_{G-U} d\mu_H \rho_t < \epsilon/2$  since the support of  $\rho_t$  gets more and more concentrated at the identity for  $t \rightarrow 0$ . Therefore we conclude that for any  $f \in C(G)$ ,  $\epsilon > 0$  there exists a  $t(\epsilon) > 0$  such that  $\|\rho_t \star f - f\|_1 < \epsilon$ .

Now let us focus on a general  $f \in L_1(G, d\mu_H)$  and consider an arbitrary  $k \in C(G)$ . Then we expand

$$\|\rho_t \star f - f\|_1 \leq \|\rho_t \star (f - k) - (f - k)\|_1 + \|\rho_t \star k - k\|_1. \quad (3.12)$$

Now  $G$  is a compact Hausdorff space and  $\mu_H$  is a Borel measure on  $G$  so that  $C(G)$  is dense in  $L_1(G, d\mu_H)$  [22]. We can therefore find a  $k$  such that  $\|f - k\|_1 < \epsilon/4$  and therefore  $\|\rho_t \star (f - k) - (f - k)\|_1 \leq 2\|f - k\|_1 \|\rho_t\|_1 \leq \epsilon/2$  for any  $t > 0$  while we have shown above that for any given  $k \in C(G)$  we can always choose  $t$  sufficiently small such that  $\|\rho_t \star k - k\|_1 < \epsilon/2$ .

This furnishes the proof.

□

The theorem can obviously be extended to functions of more than one variable. This will enable us to define the finite dimensional joint distributions of a measure.

## 3.2 Computation of the finite joint distributions of a measure

Recall [23] that every representation of a compact group is equivalent to a unitary one, so that we may restrict ourselves to unitary representations in the sequel. Also, every such representation is completely reducible. In what follows we will assume that we have fixed, in each equivalence class of irreducible representations that arise in the decomposition into irreducibles of a tensor product of irreducible representations (the ones that were fixed in definition 3.1), a standard base of independent representations which project onto orthogonal representation spaces. For the case of  $SU(2)$  this is the familiar Clebsh-Gordan decomposition and for  $GL(n)$  or  $SU(n)$  this can be established, for instance, by choosing the representations associated with the standard tableaux of the corresponding Young diagrammes [23] and for the general case we assume to have made a similar choice.

First we introduce a new notion.

**Definition 3.3** i) A loop network is a triple  $(\gamma, \vec{\pi}, \pi)$  consisting of a graph  $\gamma$ , a vector  $\vec{\pi} = (\pi_1, \dots, \pi_{n(\gamma)})$  of irreducible representations of  $G$  and an irreducible representation  $\pi$  of  $G$  which takes values in the set of irreducible representations of  $G$  contained in the decomposition into irreducibles of the tensor product  $\otimes_{k=1}^n \pi_k$ .  
ii) A loop-network state is a map from  $\overline{\mathcal{A}/\mathcal{G}}$  into  $\mathbb{C}$  defined by

$$T_{\gamma, \vec{\pi}, \pi}(A) := \text{tr}[\otimes_{k=1}^{n(\gamma)} \pi_k(h_{\beta_k(\gamma)}(A)) \cdot c(\vec{\pi}, \pi)] \quad (3.13)$$

where the matrix  $c$  is defined by

$$c(\vec{\pi}, \pi) := \sqrt{\frac{\prod_{k=1}^{n(\gamma)} d_{\pi_k}}{d_\pi}} \pi(1). \quad (3.14)$$

Loop network states satisfy the following important properties.

**Lemma 3.1** i) Given a graph  $\gamma$ , the set of all loop network states provides an orthonormal basis of  $L_2(\overline{\mathcal{A}/\mathcal{G}}_\gamma, d\mu_{0,\gamma}) = L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0) \cap \text{Cyl}_\gamma(\overline{\mathcal{A}/\mathcal{G}})$ .  
ii) Given a graph  $\gamma'$ , consider all its subgraphs  $\gamma < \gamma'$ . Remove all the loop network states on  $\gamma'$  which are pull-backs of loop-network states on  $\gamma$ . The collection of all loop-network states so obtained provides an orthonormal basis of  $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ .

Proof :

i) The orthogonality relations for loop-network states on a given graph  $\gamma$  follow easily from basic group integration theory. By the Peter&Weyl theorem we have

$$\begin{aligned} \langle T_{\gamma, \vec{\pi}, \pi}, T_{\gamma, \vec{\pi}', \pi'} \rangle &= \bar{c}(\vec{\pi}, \pi)^{(i_1, j_1), \dots, (i_n, j_n)} c(\vec{\pi}', \pi')^{(k_1, l_1), \dots, (k_n, l_n)} \times \\ &\times \delta_{\vec{\pi}, \vec{\pi}'} \frac{1}{\prod_{k=1}^n d_{\pi_k}} \delta_{i_1, k_1} \delta_{j_1, l_1} \cdots \delta_{i_n, k_n} \delta_{j_n, l_n} \\ &= \delta_{\vec{\pi}, \vec{\pi}'} \frac{1}{\sqrt{d_\pi d_{\pi'}}} \text{tr}[\pi^\dagger(1) \pi'(1)] \\ &= \delta_{\vec{\pi}, \vec{\pi}'} \delta_{\pi, \pi'} \end{aligned} \quad (3.15)$$

where we have used that the non-equivalent irreducible – as well as our choice of equivalent – representations of a compact gauge groups are orthogonal, that is,  $\pi(1)$  is a projector.

The completeness of these states on  $L_2(\overline{\mathcal{A}/\mathcal{G}}_\gamma, d\mu_{0,\gamma})$  follows also from the Peter&Weyl theorem together with a gauge-invariance argument :

We know that the states

$$T_{\gamma, \vec{\pi}}^{(i_1, j_1) \dots (i_n, j_n)} := [\otimes_{k=1}^n \pi_k(h_{\beta_k(\gamma)}(A))]^{(i_1, j_1) \dots (i_n, j_n)} \quad (3.16)$$

contain an overcomplete set of states for  $L_2(\overline{\mathcal{A}/\mathcal{G}}_\gamma, d\mu_{0,\gamma})$  and thus we only need to select all the independent gauge invariant combinations of those, that is, we need to find all the matrices  $c$ , called contractors, which turn (3.16) into gauge invariant states when being contracted with them.

Notice that all the generators  $\beta$  are based loops. Therefore under a gauge transformation

$$\text{tr}[T_{\gamma, \vec{\pi}} \cdot c] \rightarrow \text{tr}[T_{\gamma, \vec{\pi}} \cdot (\otimes_{k=1}^n \pi_k)(g^{-1}) \cdot c \cdot (\otimes_{k=1}^n \pi_k)(g)] \quad (3.17)$$

and gauge invariance requires choosing  $c$  such that

$$(\otimes_{k=1}^n \pi_k)(g^{-1}) \cdot c \cdot (\otimes_{k=1}^n \pi_k)(g) = c \text{ for all } g \in G. \quad (3.18)$$

Now notice that the matrix  $c$  in (3.17) is already projected on the reducible representation space defined by the tensor product representation  $\otimes_{k=1}^n \pi_k$  because the matrix  $(\otimes_{k=1}^n \pi_k)(1)$  is a projector on that space and leaves the matrix  $T_{\gamma\vec{\pi}}$  in (3.16) invariant under multiplication from both sides. It follows that we can expand

$$c = \sum_{\pi \in \otimes_{k=1}^n \pi_k} c_\pi \quad (3.19)$$

where the sum is over the irreducibles contained in the decomposition of  $\otimes_{k=1}^n \pi_k$  into irreducibles and the matrix  $c_\pi$  is projected onto the representation space labelled by  $\pi$ , namely  $\pi(1)c_\pi = c_\pi\pi(1) = c_\pi$ . Let us also decompose

$$(\otimes_{k=1}^n \pi_k(g)) = \sum_{\pi \in \otimes_{k=1}^n \pi_k} \pi(g) \quad (3.20)$$

We now plug (3.19) and (3.20) into (3.18) and obtain

$$\sum_{\pi \in \otimes_{k=1}^n \pi_k} \pi(g) c_\pi \pi(g)^{-1} = \sum_{\pi \in \otimes_{k=1}^n \pi_k} c_\pi \quad (3.21)$$

which we multiply by  $\pi(1)$  to obtain

$$\pi(g) c_\pi \pi(g)^{-1} = c_\pi, \quad (3.22)$$

that is,  $c_\pi$  commutes with the representation and therefore must be proportional to  $\pi(1)$  by the lemma of Schur.

ii) It follows immediately from i) that the union of all the loop-network states for all the graphs  $\gamma$  is an overcomplete (uncountable) set of states on  $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$  (the graphs label cylindrical functions which are dense in  $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ , compare also [19]). The redundant states are eliminated by the recipe stated in the lemma. In particular then all the representations involved in  $\vec{\pi}$  are required to be non-trivial (except for the empty graph) since any loop-network with trivial representations can be realized already on a smaller graph.

It remains to show that then two loop-network states that are defined on different graphs are orthogonal. But this is trivial because for two graphs  $\gamma \neq \gamma'$  there is at least one generator  $\beta$  in which they differ and the representation  $\pi$  associated with that generator is non-trivial. Therefore the inner product between these loop-network states will contain the integral  $\int d\mu_H(g) \pi(g) = 0$ . Therefore we get altogether

$$\langle T_{\gamma, \vec{\pi}, \pi}, T_{\gamma', \vec{\pi}', \pi'} \rangle = \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \delta_{\pi, \pi'}. \quad (3.23)$$

□

Remark :

More concretely, the redundant states can be removed by imposing the following constraints on the vector  $\vec{\pi}$  : given an edge  $e$  of  $\gamma$  (that is, a maximally analytic piece of  $\gamma$ ) determine the generators, say  $\beta_1, \dots, \beta_k$ , which contain  $e$ . If  $\beta_i$  is cloured



with the representation  $\pi_i$  then require that the tensor product  $\pi_1 \otimes \dots \otimes \pi_k$  does not contain any trivial representation otherwise the loop-network state would contain a piece defined on a smaller graph.

This restriction leads to the definition of edge-network states (compare the appendix).

The next thing to do is to define the Fourier transform of a measure on  $\overline{\mathcal{A}/\mathcal{G}}$ .

**Definition 3.4** *The loop transform (Fourier transform, characteristic functional) of a measure  $\mu$  on  $\overline{\mathcal{A}/\mathcal{G}}$  is defined by*

$$\chi_\mu(\gamma, \vec{\pi}, \pi) := \langle \bar{T}_{\gamma, \vec{\pi}, \pi} \rangle := \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu(A) \bar{T}_{\gamma, \vec{\pi}, \pi}(A) \quad (3.24)$$

This definition differs from the one given in [1, 10], however, both definitions are equivalent in the sense that they allow for a reconstruction of  $\mu$  according to the Riesz-Markov theorem [22]. Namely, the former definition is based on the vacuum expectation value of products of Wilson loop functionals, and according to [1, 24], these functions are an overcomplete set of functions on  $\overline{\mathcal{A}/\mathcal{G}}$  (that is, they are subject to Mandelstam identities) so that we can reexpress them in terms of loop networks and vice versa.

Now let be given a functional  $\chi$  on loop-networks. Provided it is positive (note that there are no Mandelstam relations between loop network states any more and that the product of loop network states is a linear combination of loop network states) we know by the Riesz-Markov theorem that there is a measure  $\mu$  whose Fourier transform is given by  $\chi$ . This measure will be known if we know its finite joint distributions which automatically form a self-consistent system of measures whose projective limit (known to exist) gives us back  $\mu$ . We now compute these joint distributions.

**Lemma 3.2** *If the Fourier transform of a (complex) regular Borel measure  $\mu$  on a compact gauge group  $G$  is in  $\ell_1$  then it is absolutely continuous with respect to the Haar measure on  $G$ .*

Proof :

Given the Fourier coefficients  $\chi_\pi^{ij}$  of the measure  $\mu$  the Fourier series  $\hat{\chi}$  associated with these coefficients is an  $L_1(G, d\mu_H)$  function due to the anticipated  $\ell_1$  property of the Fourier coefficients. Moreover, the measure  $d\hat{\mu}(g) := \hat{\chi}(g) d\mu_H(g)$  has the same Fourier transform as  $\mu$ . Since the functions defined by finite linear combinations of the functions  $\sqrt{d_\pi} \pi_{ij}$  form a dense set in  $C(G)$ ,  $G$  being a compact Hausdorff group, it follows that both measures define the same bounded linear functional. Now we infer from the uniqueness part of the Riesz-Markov theorem that indeed  $\mu = \hat{\mu}$  from which absolute continuity follows. If  $\mu$  is even a positive measure then  $\hat{\chi}$  is positive.  $\square$

The theorem can obviously extended to any finite number of variables.

**Theorem 3.3** *Let  $\chi$  be a positive linear functional on  $C(\overline{\mathcal{A}/\mathcal{G}})$ . Then  $\chi$  is the loop transform of a positive regular Borel measure  $\mu$  on  $\overline{\mathcal{A}/\mathcal{G}}$ . If for a given graph  $\gamma$*

with  $n$  generators the sequence  $\{\chi(\gamma, \vec{\pi}, \pi) \sqrt{d_\pi \prod_{k=1}^n d_{\pi_k}}\}$  is in  $\ell_1$  then the finite joint distributions of  $\mu_\gamma$  are (in the sense of  $L_1(\mathcal{A}/\mathcal{G}_\gamma, \mu_{0,\gamma})$ ) given by

$$\frac{d\mu_\gamma(A)}{d\mu_{0,\gamma}(A)} = \sum_{\vec{\pi}} \sum_{\pi \in \otimes_{k=1}^n \pi_k} \chi(\gamma, \vec{\pi}, \pi) T_{\gamma, \vec{\pi}, \pi}(A) . \quad (3.25)$$

Proof :

The proof is a straightforward application of the inverse Fourier transform, theorem 3.2.

The convergence condition on the characteristic functional mentioned in the theorem together with lemma 3.2 allows us to conclude that on cylindrical subspaces the measure  $\mu_\gamma$  is absolutely continuous with respect to the induced Haar measure  $\mu_{0,\gamma}$ . Therefore there exists a positive  $L_1(G^n, d^n \mu_H)$  function  $\rho_\gamma(g_1, \dots, g_n)$  such that it is the Radon-Nikodym derivative of  $d\mu_\gamma$  with respect to  $d\mu_{0,\gamma}$  [22]. Since  $\mu$  is a gauge-invariant measure, the Fourier coefficients of  $\rho_\gamma$  satisfy

$$(\otimes_{k=1}^n \pi_k(g^{-1}))_{(i_1, k_1) \dots (i_n, k_n)} \rho_{\gamma, \vec{\pi}}^{(k_1, l_1) \dots (k_n, l_n)} (\otimes_{k=1}^n \pi_k(g))_{(l_1, j_1) \dots (l_n, j_n)} = \rho_{\gamma, \vec{\pi}}^{(i_1, j_1) \dots (i_n, j_n)} \quad (3.26)$$

so they lie in the invariant subspace spanned by the matrices  $\pi(1)$  where  $\pi \in \otimes_{k=1}^n \pi_k$ . Thus

$$\begin{aligned} \rho_{\gamma, \vec{\pi}}^{(i_1, j_1) \dots (i_n, j_n)} &= \sum_{\pi \in \otimes_{k=1}^n \pi_k} \frac{1}{d_\pi} \text{tr}[\rho_{\gamma, \vec{\pi}} \pi(1)] \pi(1)_{(i_1, j_1) \dots (i_n, j_n)}(1) \\ &= \sum_{\pi \in \otimes_{k=1}^n \pi_k} \frac{1}{\sqrt{d_\pi}} \chi(\gamma, \vec{\pi}, \pi) \pi_{(i_1, j_1) \dots (i_n, j_n)}(1) . \end{aligned} \quad (3.27)$$

Now

$$\begin{aligned} &\sum_{i_1, j_1=1}^{d_{\pi_1}} \sqrt{d_{\pi_1}} \dots \sum_{i_n, j_n=1}^{d_{\pi_n}} \sqrt{d_{\pi_n}} |\rho_{\gamma, \vec{\pi}}^{(i_1, j_1) \dots (i_n, j_n)}| \\ &\leq \sum_{\pi \in \otimes_{k=1}^n \pi_k} |\chi(\gamma, \vec{\pi}, \pi)| \sum_{i_1, j_1=1}^{d_{\pi_1}} \dots \sum_{i_n, j_n=1}^{d_{\pi_n}} |c(\vec{\pi}, \pi)_{(i_1, j_1) \dots (i_n, j_n)}(1)| \\ &= \sum_{\pi \in \otimes_{k=1}^n \pi_k} |\chi(\gamma, \vec{\pi}, \pi)| \sqrt{d_\pi \prod_{k=1}^n d_{\pi_k}} \end{aligned} \quad (3.28)$$

Therefore the convergence condition on  $\chi$  mentioned in the theorem implies that  $\hat{\rho}_\gamma \in \ell_1$  and the theorem on the inverse Fourier transform tells us that in the sense of  $L_1$

$$\begin{aligned} \rho_\gamma(g_1, \dots, g_n) &= \sum_{\vec{\pi}} \sum_{i_1, j_1=1}^{d_{\pi_1}} \sqrt{d_{\pi_1}} \dots \sum_{i_n, j_n=1}^{d_{\pi_n}} \sqrt{d_{\pi_n}} \times \\ &\times \rho_{\gamma, \vec{\pi}}^{(i_1, j_1) \dots (i_n, j_n)} (\otimes_{k=1}^n \pi_k)_{(i_1, j_1) \dots (i_n, j_n)}(g_1, \dots, g_n) \\ &= \sum_{\vec{\pi}} \sum_{\pi \in \otimes_{k=1}^n \pi_k} \chi(\gamma, \vec{\pi}, \pi) T_{\gamma, \vec{\pi}, \pi} \end{aligned} \quad (3.29)$$

where we used (3.27) and the definition of a loop-network. This furnishes the proof.

□

Note that if we knew that  $\rho_\gamma \in L_2(\overline{\mathcal{A}/\mathcal{G}}_\gamma, d\mu_{0,\gamma})$  then we could have simply made use of the fact that loop networks provide for an orthonormal basis of  $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$  to conclude theorem 3.3 directly from the gauge invariant version of the Peter&Weyl theorem. This is, however, not necessarily the case.

## 4 Examples of inverse Fourier transforms

In order to determine whether a given function  $\chi$  from (singular) knots into the complex numbers arises as the loop transform of a measure one has to check two things :

- 1) All the identities that are satisfied by products of traces of holonomies of loops have to be satisfied Mandelstam identities [24]. Alternatively, it has to be true that  $\chi$  can be written purely in terms of loop-network states.
- 2) It is a positive linear functional on any cylindrical subspace of  $C(\overline{\mathcal{A}/\mathcal{G}})$ .
  - a) An example of a (singular) knot function that satisfies these criteria is of course the Fourier transform of any  $\sigma$ -additive measure on  $\overline{\mathcal{A}/\mathcal{G}}$ . Let us look at the Fourier transform of the measure  $\mu_0$  which is even diffeomorphism invariant so that  $\chi$  is a singular knot invariant :

$$\chi_{\mu_0}(\gamma, \vec{\pi}, \pi) = \delta_{\vec{\pi}, 0} \delta_{\pi, 0} \quad (4.1)$$

where 0 denotes the trivial representation. In other words,  $\chi$  is non-vanishing only on the trivial loop network 1. Therefore we find for the finite joint distribution precisely  $\rho_\gamma(g_1, \dots, g_n) = 1$  according to (3.25).

- b) A second example of a singular knot invariant is given as follows :

Let  $K$  be a regular knot invariant and let  $\text{orb}(\alpha_0)$  be the orbit of the regular knot  $\alpha_0$  under the diffeomorphism group under question. Then the formal expression

$$d\mu(A) := \sum_{\alpha \in \text{orb}(\alpha_0)} \bar{T}_\alpha(A) d\mu_0(A) \quad (4.2)$$

is a diffeomorphism invariant measure on  $\overline{\mathcal{A}/\mathcal{G}}$  where  $T_\alpha = \text{tr} h_\alpha$ , namely its Fourier transform

$$\chi_\mu(\gamma, \vec{\pi}, \pi) = \chi_{\text{orb}(\alpha_0)}(\gamma) \delta_{\vec{\pi}, \text{def}} \delta_{\pi, \text{def}} \quad (4.3)$$

( $\chi_S$  means the characteristic function of a set  $S$ ) is a bounded linear functional on  $C(\overline{\mathcal{A}/\mathcal{G}})$  and thus by the extension of the Riesz-Markov theorem [22] we know that it corresponds to a unique complex regular Borel measure on  $\overline{\mathcal{A}/\mathcal{G}}$  which is rigorously defined. So we get a new singular knot invariant from a regular one !

- c) Now we provide an example of a Fourier transform for a non-diffeomorphism invariant measure :

In two Euclidean spacetime dimensions one can choose the generators of a graph to be simple, meaning that they do not have self-intersections, and non-overlapping, meaning that the intersection of the surfaces that any two of them enclose have zero Euclidean area [11]. Then the characteristic functional (in the continuum) for pure Yang-Mills theory on the Euclidean plane is given by

$$\chi_{\mu_{YM}^{(2)}}(\gamma, \vec{\pi}, \pi) = e^{-\frac{g_0^2}{2} \sum_{k=1}^n \lambda_{\pi_k} \text{Ar}(\beta_k)} \sqrt{d_\pi \prod_{k=1}^n d_{\pi_k}} \quad (4.4)$$

where  $g_0$  is the bare coupling constant,  $-\lambda_\sigma$  is the eigenvalue of the Casimir operator on the matrix element functions  $\sigma_{ij}(g)$  for an irreducible representation  $\sigma$  and  $\text{Ar}(\alpha)$  is the area of the surface enclosed by a simple loop  $\alpha$ .

Let us compute, for instance, the one-dimensional joint distributions. We find from (3.25) (if  $\vec{\pi}$  is one dimensional then of course also  $\pi = \vec{\pi}$  is the only possible choice)

$$\rho_{\gamma=\beta}(g) = \sum_{\pi} d_{\pi} e^{-\frac{g_0^2}{2} \text{Ar}(\beta) \lambda_{\pi}} \chi_{\pi}(g) \quad (4.5)$$

therefore  $\rho_{\gamma=\beta}(g) = \rho_{g_0^2 \text{Ar}(\beta)}(g)$  where, as before,  $\rho_t$  is the heat kernel on  $G$ .

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## A Edge networks

In this appendix we show how the developments of this paper can be written in terms of edges which are more convenient to deal with if

- a) one wants to write down a complete orthonormal basis [18, 19] of  $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$  without having to make use of the recipe mentioned in lemma 3.1,ii) and
- b) if one is interested in applications to quantum gravity, in particular if one is to obtain the spectrum of certain area and volume operators [12, 25].

Again we consider an unoriented graph  $\gamma$  and fix an orientation for each of its edges once and for all. Note that we do not have to make a choice of edges in this case because they are defined to be the maximally analytic pieces of the given graph. Denote by  $E_{\gamma}$  and  $V_{\gamma}$  the set of edges and vertices of  $\gamma$  respectively,  $n_E(\gamma)$  and  $n_V(\gamma)$  are the number of these edges and vertices respectively and in general we will denote edges by the symbol  $e$  and vertices by the symbol  $v$ .

**Definition A.1** *i) An edge network is a triple  $(\gamma, \vec{\pi}, \vec{\sigma})$  consisting of a graph  $\gamma$  an edge vector of irreducible non-trivial representations  $\vec{\pi} = (\pi_1, \dots, \pi_{n_E(\gamma)})$  and a vertex vector of irreducible trivial representations  $\vec{\sigma} = (\sigma_1, \dots, \sigma_{n_V(\gamma)})$ . The irreducible representation  $\sigma_v$  takes values in the set of trivial irreducible representations that are contained in the decomposition into irreducibles of  $\otimes_{e^-=v} \vec{\pi}_e \otimes_{e^+=v} \pi_e$ ,  $e^{\pm}$  being the starting or ending point of  $e$  (these representations are automatically orthogonal to each other). We assume that  $\vec{\pi}$  is such that the space of possible  $\vec{\sigma}$  is non-empty.*  
*ii) An edge network state is a function from  $\overline{\mathcal{A}/\mathcal{G}}$  into the complex numbers defined by*

$$T_{\gamma, \vec{\pi}, \vec{\sigma}}(A) := \text{tr}[\otimes_{e \in E_{\gamma}} \pi_e(h_e(A)) \cdot c(\gamma, \vec{\pi}, \vec{\sigma})] \quad (\text{A.1})$$

where the matrix  $c$  is defined as follows :

There exist permutation matrices  $P_{\gamma}^{\pm}$  such that

$$(P_{\gamma}^{\pm})^{-1} \cdot \otimes_{e \in E_{\gamma}} \pi_e(h_e) \cdot P_{\gamma}^{\pm} = \otimes_{v \in V_{\gamma}} \otimes_{e^{\pm}=v} \pi_e(h_e) \quad (\text{A.2})$$

then

$$c(\gamma, \vec{\pi}, \vec{\sigma}) := \sqrt{\prod_{e \in E_\gamma} d_{\pi_e} P_\gamma^-} \cdot \otimes_{v \in V_\gamma} c_v(\vec{\sigma}) \cdot (P_\gamma^+)^{-1} \quad (\text{A.3})$$

where the vertex contractor is given by

$$c_v(\vec{\sigma})_{ij} := \frac{\sigma_v(1)_{(i_0 j_0), (ij)}}{\sqrt{\sigma_v(1)_{(i_0 j_0), (i_0 j_0)}}} \quad (\text{A.4})$$

and where  $(i_0 j_0)$  is an arbitrary but fixed choice of index pairs such that the denominator of (A.4) is non-vanishing (since  $\sigma_v$  is a trivial representation, any choice of  $i_0, j_0$  leads to the same vector  $c_v$  up to a multiple constant). Here the index structure comes from  $[\otimes_{e^- = v} \pi_e] \otimes [\otimes_{e^+ = v} \pi_e]_{(ij), (i_0 j_0)}$ .

The composition of loops can result in a loop that is defined already on a smaller graph. This is the source of the redundancy mentioned in lemma 3.1. Something similar cannot happen with edges whence there is no redundancy in the definition of edge networks.

In the special case of  $G = SU(2)$  the notion of edge-networks coincides with the notion of spin-networks [18, 19]. In this case the vertex contractors can easily be seen to be the usual Clebsh-Gordan coefficients  $c_v = \langle 0, \{(j_v, l_v)\} \rangle$  for the addition of all the angular momenta corresponding to the irreducible representations with which the edges starting and ending at  $v$  are coloured. Note that our analytical expression (A.1) for an edge-network state does not need any graphical visualization and no preferred role is played by trivalent graphs [18].

**Theorem A.1** *The set of all edge networks provides for a complete orthonormal basis of  $L_2(\mathcal{A}/\mathcal{G}, d\mu_0)$ .*

Proof :

1) Orthonormality :

Note that  $P_\gamma^T = P_\gamma^\dagger = P_\gamma^{-1}$  for any permutation matrix  $P_\gamma$  which merely reshuffles the order of the factors in the tensor product  $\otimes_{e \in E_\gamma} \pi_e$ . Therefore

$$\begin{aligned} \langle T_{\gamma, \vec{\pi}, \vec{\sigma}}, T_{\gamma', \vec{\pi}', \vec{\sigma}'} \rangle &= \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \text{tr}[c(\gamma, \vec{\pi}, \vec{\sigma})^\dagger c(\gamma, \vec{\pi}, \vec{\sigma})] \\ &= \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \text{tr}[(P_\gamma^- \cdot \otimes_{v \in V_\gamma} c_v(\vec{\sigma}) \cdot (P_\gamma^+)^{-1})^\dagger \cdot \\ &\quad P_\gamma^- \cdot \otimes_{v \in V_\gamma} c_v(\vec{\sigma}') \cdot (P_\gamma^+)^{-1}] \\ &= \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \text{tr}[\otimes_{v \in V_\gamma} (c_v(\vec{\sigma}))^\dagger c_v(\vec{\sigma}')] \\ &= \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \delta_{\vec{\sigma}, \vec{\sigma}'} \end{aligned} \quad (\text{A.5})$$

we have orthonormality. We have used that two different graphs differ in at least one edge which carries a non-trivial irreducible representation and therefore the integral (A.5) with respect to the Haar measure vanishes as well as the reality and symmetry of the projectors  $\pi(1)$  for any irreducible representation  $\pi$ .

2) Completeness :

As in (3.16) we start from the observation that the states

$$(T_{\gamma, \vec{\pi}})_{(i_1 j_1) \dots (i_{n(E_\gamma)} j_{n(E_\gamma)})} = (\otimes_{e \in E_\gamma} \pi_e(h_e))_{(i_1 j_1) \dots (i_{n(E_\gamma)} j_{n(E_\gamma)})} \quad (\text{A.6})$$

contain a complete set of states for  $L_2(\overline{\mathcal{A}/\mathcal{G}}_\gamma, d\mu_{0,\gamma})$  so that we only need to contract (A.6) in all the possible gauge invariant ways. Making use of (A.2) we see that under a gauge transformation  $g_v$  at every vertex  $v$  we have

$$\begin{aligned} \text{tr}[T_{\gamma,\vec{\pi}} \cdot c] &\rightarrow \text{tr}[T_{\gamma,\vec{\pi}} \cdot P_\gamma^- \cdot [\otimes_v \otimes_{e^-=v} \pi_e(g_v^{-1})] \cdot \\ &[(P_\gamma^-)^{-1} \cdot c \cdot P_\gamma^+] \cdot [\otimes_v \otimes_{e^+=v} \pi_e(g_v)] \cdot (P_\gamma^+)^{-1}] \end{aligned} \quad (\text{A.7})$$

so that gauge invariance requires that

$$[\otimes_v \otimes_{e^-=v} \pi_e(g_v^{-1})] \cdot [(P_\gamma^-)^{-1} \cdot c \cdot P_\gamma^+] [\otimes_v \otimes_{e^+=v} \pi_e(g_v)] = [(P_\gamma^-)^{-1} \cdot c \cdot P_\gamma^+] \cdot (\text{A.8})$$

Without loss of generality we write  $c = P_\gamma^- \cdot \sum_{i=1}^n \otimes_v s_v^{(i)} \cdot (P_\gamma^+)^{-1}$  for some  $n$  and arrive at

$$\sum_i \otimes_v s_v^{(i)} = \sum_i \otimes_v [\otimes_{e^-=v} \pi_e(g_v^{-1})] \cdot s_v^{(i)} \cdot [\otimes_{e^+=v} \pi_e(g_v)] \quad (\text{A.9})$$

which we can write more conveniently as the eigenvalue equation

$$\sum_i \otimes_v s_v^{(i)} = [\sum_i \otimes_v s_v^{(i)}] \cdot [\otimes_v \nu_v(g_v)] \quad (\text{A.10})$$

where we have abbreviated

$$\nu_v(g_v) = [\otimes_{e^-=v} \bar{\pi}_e \otimes_{e^+=v} \pi_e](g_v). \quad (\text{A.11})$$

Since  $\nu_v(1)$  is a projector, it follows that the eigenvector on the lhs of (A.10) lies in the (reducible) subspace corresponding to  $\otimes_v \nu_v$ , that is, each  $s_v^{(i)}$  lies in the (reducible) subspace corresponding to  $\nu_v$ .

We now ask for the possible solutions  $x$  of the equation  $x[\otimes_v \nu_v(g_v)] = x \forall (g_v)_{v \in V_\gamma} \in G^{n(V_\gamma)}$ . Setting  $g_{v'} = 1$  for all  $v' \in V_\gamma$  except for one,  $v$  say, shows that  $x$  has to project onto one of the trivial representations  $\sigma_v$  contained in  $\nu_v$ . Therefore  $x$  has the general form  $\sum_{\vec{\sigma} \in \vec{\nu}=\text{trivial}} k_{\vec{\sigma}} \otimes_v c_v(\vec{\sigma})$  for some complex numbers  $k_{\vec{\sigma}}$  which is precisely of the form (A.4). Since the space of vectors spanned by the matrix elements of the trivial representation is one-dimensional the choice made in (A.4) means no loss of generality. This proves completeness.

□

It is straightforward to see how the inverse loop transform theorem can be translated into an “inverse edge transform” theorem.

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